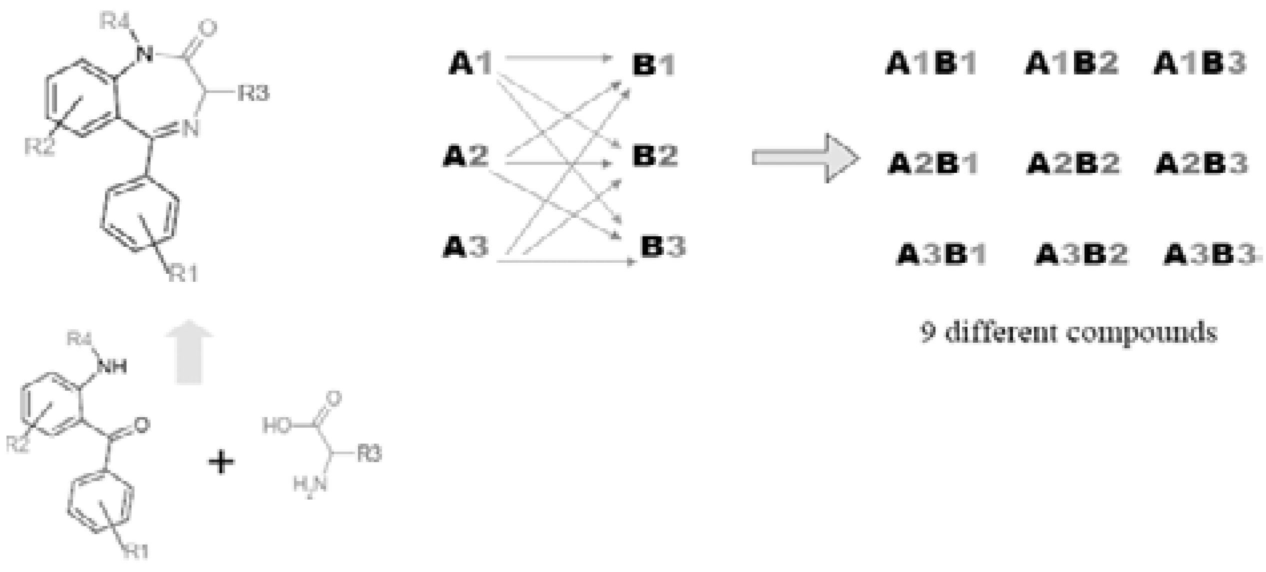
It is a similar concept here with combinatorial chemistry. By predicting the colossal varieties of compounds that can be formed from a reaction, a large amount of useful yet slightly different compounds can be obtained from a single reaction. (Refer to the figure below.) This is particularly important for pharmaceutical companies who wish to manufacture a number of similar compounds quickly and cheaply.

In practice, huge 'libraries' containing every feasible configuration of the resulting compounds are created using combinatorial methods. Computer analysis is then carried out to find out which compounds are useful and appropriate separation methods are designed.

For other applications, chemists use combinatorial chemistry to generate large quantities of different compounds quickly (to increase chemical diversity) and test them with a single reagent (e.g. enzyme solution) to see which compounds fit into the enzymes' active sites. It is now being used to shorten the time needed to develop new drugs, which is especially important with the current increase in deadly antibiotic-resistant pathogens.

These are just the tip of the iceberg when it comes to mathematical applications in real life. I could go on and talk about how mathematics plays a part in human behaviour, or how it could possibly trigger cellular division, but these are all intriguing questions left for the reader to explore. Mathematics is not just a 'universal language' or a rigorous subject about proofs and abstract symbols; it is also one of exciting applications and groundbreaking discoveries. The next time you flip through your science or chemistry textbooks, think about the mathematics behind each and every statement – you'd be surprised by just how useful it could be.



Here, R1 to R4 denote different substituents (additional chains of atoms).

Regiomontanus' Angle Maximization

A Classical Construction Approach

Li Kwing Hei 11D

You have been waiting for ages for this day - Valentine's Day! Your eyes gleam with joy as you visualize taking the love of your life to the cinema for a movie. You want everything to be perfect. After all, you are going to propose to her tonight.

But wait. Which seats should you pick for the movie? Your palms grow sweaty as you ponder this challenging problem. Where should you sit such that the movie screen appears largest? The middle column obviously, but which row? Apparently, it is time for Mathematics to save the day.

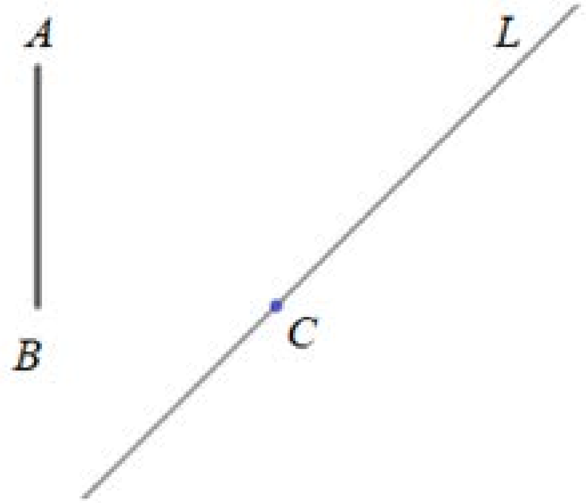


Figure 0.1

Story aside, let's construct a diagram for the situation. In Figure 0.1, A and B represent the top and the bottom of the movie screen respectively. Denote any chosen column of seats by L and your seat by C on L. Since you would like the movie screen to appear the largest, you want to locate C on L such that $\angle ACB$ is largest.

Indeed, this problem was first posed in the 15th century by a German mathematician called Johannes Müller. Johannes Müller was not interested in finding the best seats with his girlfriend in a cinema, but he was curious to find the best place to stand to observe a painting. To tackle this problem, there are a number of ways, many of which involve complicated trigonometric functions. Since half-angle formulas and differentiation may be slightly intimidating, we shall solve the problem with classical construction.

Part 1: The Basics

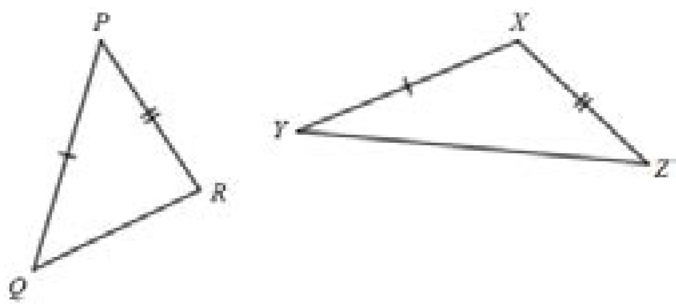


Figure 1.1

Before we dive straight into the solution, we shall mention some basic results in geometry. In Figure 1.1, $PQ = XY$ and $PR = XZ$. If $\angle QPR < \angle YXZ$, which side is longer, QR or YZ? It is obvious that YZ is longer. In fact, this can be proved with the cosine law. Note that

$$QR^2 = PQ^2 + PR^2 - 2(PQ)(PR)\cos\angle QPR$$

$$YZ^2 = XY^2 + XZ^2 - 2(XY)(XZ)\cos\angle YXZ$$

Since $\cos x°$ is a decreasing function on the interval $[0, 180]$, $\cos\angle QPR > \cos\angle YXZ$. It follows from the above equations that $QR^2 < YZ^2$ and $QR < YZ$. This is commonly known as the Hinge theorem. The converse of the Hinge theorem also holds: if $QR < YZ$, then $\angle QPR < \angle YXZ$.

Let's have a look at another useful result. In Figure 1.2, STV is a circle. ST is produced to U such that UV is tangent to the circle at V.

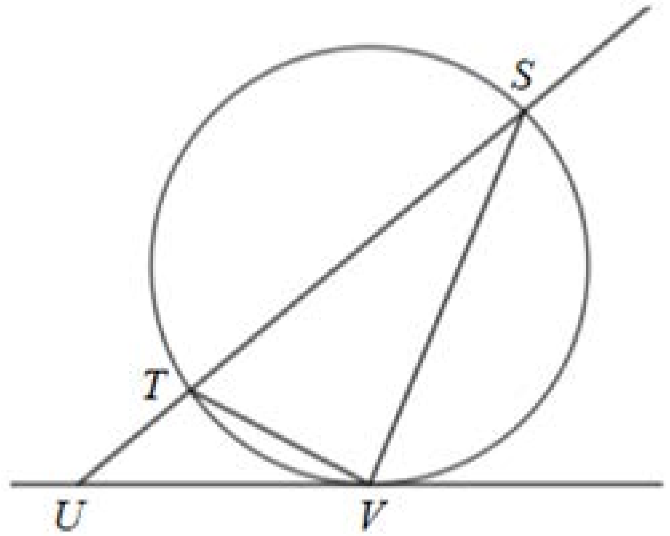


Figure 1.2

Consider ΔTUV and ΔVUS.

$\angle TUV = \angle SUV$ (common $\angle$)

$\angle TVU = \angle USV$ ($\angle$ in alt. segment)

$\Delta TUV \sim \Delta VUS$ (AA)

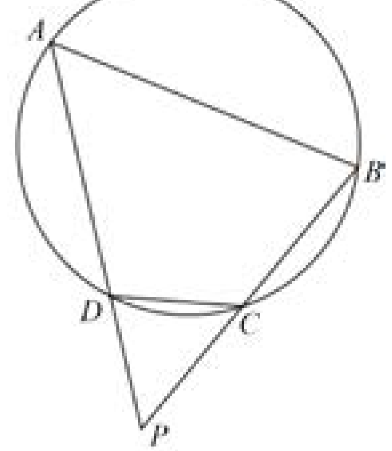
$\dfrac{TU}{VU} = \dfrac{UV}{US}$ (corr. sides, ~Δs)

$UT \cdot US = UV^2$

The common value in the last equation is known as the power of U with respect to the circle STV. Conversely, if $UT \cdot US = UV^2$ and T is a point of internal division of US, then a circle passing through S, T and V can be constructed such that UV is tangent to the circle at V.

Try it out!

In the figure, $ABCD$ is a quadrilateral inscribed in the circle. AD and BC are extended to meet at P. Can you prove that $AP \cdot DP = BP \cdot CP$?



The proof can be found on Page 7.

Part 2: Maximizing Angles with Circles

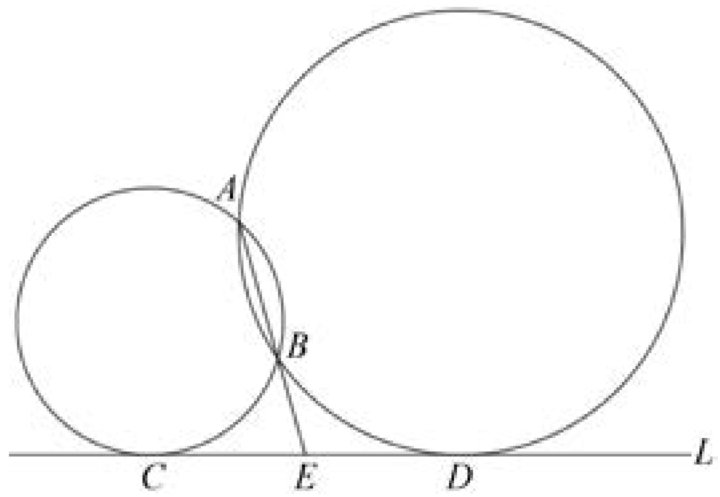


Figure 2.1

In Figure 2.1, A and B are two points above a straight line L. AB is produced to meet L at E. Two circles passing through A and B are drawn to touch L. L touches the left circle and the right circle at C and D respectively. C lies on the left of E and D lies on the right of E. Let F be a moving point on L. Suppose F is on the left of E and AF cuts the left circle at G, as shown in Figure 2.2.

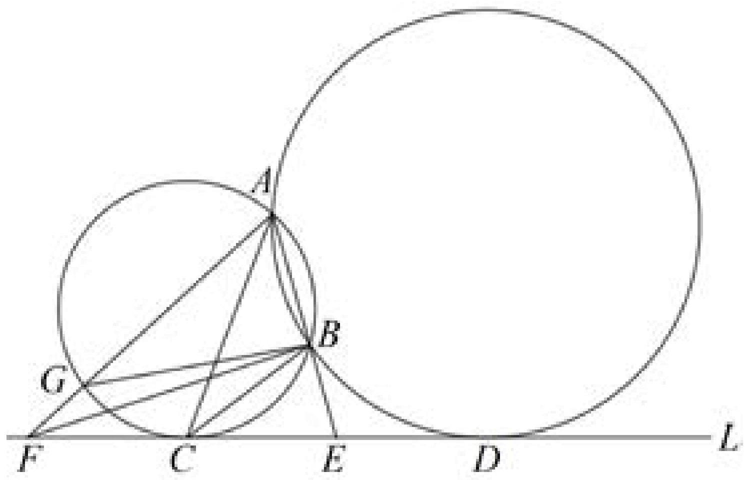


Figure 2.2

Consider ΔBFG.

$\angle AFB + \angle FBG = \angle AGB$ (ext. $\angle$ of Δ)

$\angle AGB > \angle AFB$

$\angle AGB = \angle ACB$ ($\angle$s in the same segment)

$\angle ACB > \angle AFB$

But what if F lies on the right of E? Suppose F is on the right of E and AF cuts the right circle at G, as shown in Figure 2.3.

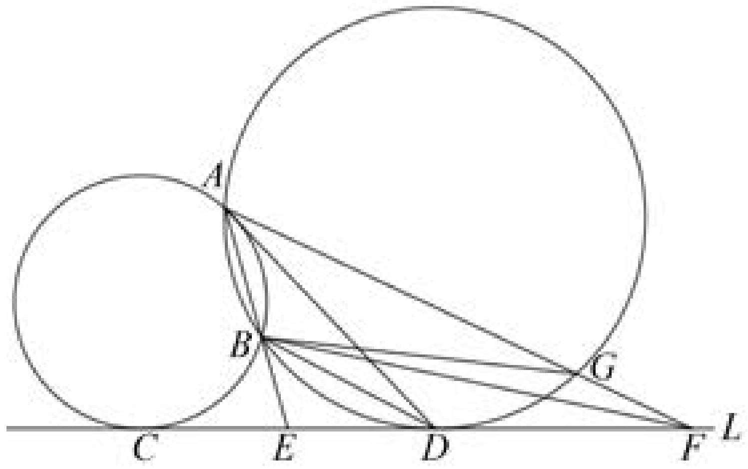


Figure 2.3

Consider ΔBFG.

$\angle AFB + \angle FBG = \angle AGB$ (ext. $\angle$ of Δ)

$\angle AGB > \angle AFB$

$\angle AGB = \angle ADB$ ($\angle$s in the same segment)

$\angle ADB > \angle AFB$

Combining the cases, as F moves along L, $\angle AFB$ is maximum when F is at either C or D. Next, we shall locate C and D, the points of tangency between L and the two circles passing through A and B, by constructing these two circles which touch L. This geometric construction problem is known as Apollonius' problem with two points and a line.

Part 3: Apollonius' Problem with Two Points and a Line

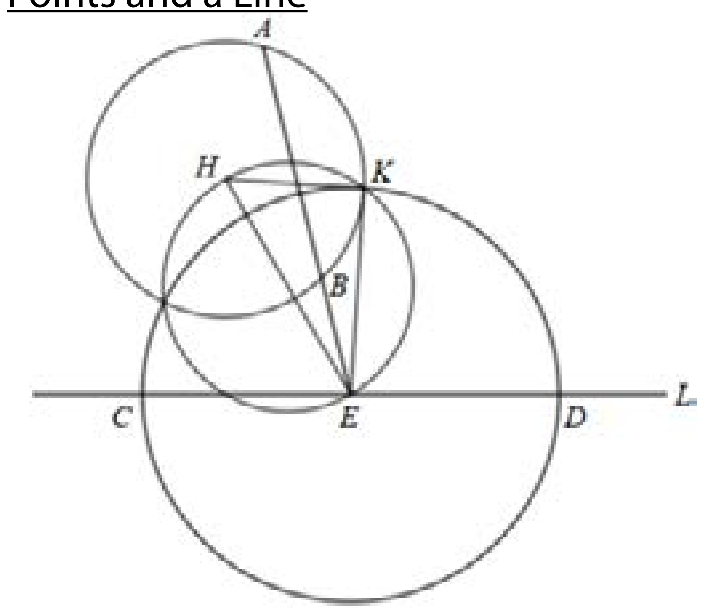


Figure 3.1

In Figure 3.1, A and B are two points above L. We want to construct two circles passing through A and B to touch L. How could that be done?

To solve the problem, AB is extended to meet L at E. Construct any circle passing through A and B. (For example, the circle with diameter AB can be chosen.) Let the centre of the circle be H and construct a circle with diameter HE. Denote any of the two points of intersection of the two circles by K. Then $\angle EKH = 90°$ and EK is tangent to the circle ABK at K. Considering the power of E with respect to the circle ABK, we have $EB \cdot EA = EK^2$. Finally, construct a circle centred at E with radius EK to intersect L at C and D. Since $EC = EK$, we have $EB \cdot EA = EC^2$. It follows that a circle passing through A, B and C can be constructed such that EC is tangent to the circle ABC at C. Similarly, a circle passing through A, B and D can be constructed such that ED is tangent to the circle ABD at D.

It has been shown in Part 2 that $\angle AFB$ is maximum when F is at either C or D. After locating C and D, it remains to compare $\angle ACB$ and $\angle ADB$.

Part 4: Comparing the Two Angles

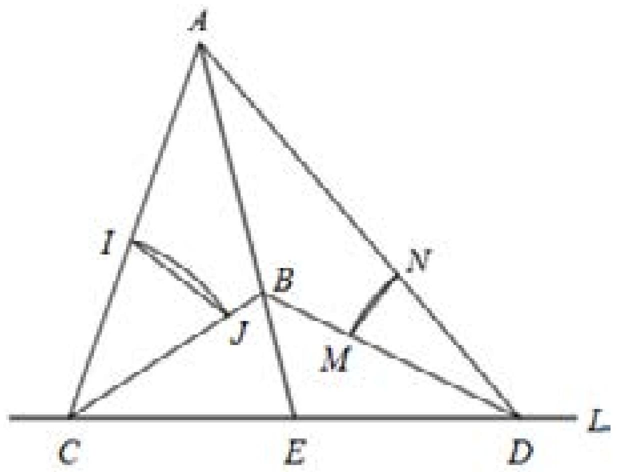


Figure 4.1

There are many ways to compare two angles, and we shall use one of the simplest. In Figure 4.1, C and D are the centres of the two arcs of equal radii. The arc centred at C cuts AC and BC at I and J respectively while the arc centred at D cuts BD and AD at M and N respectively. Hence $CI = DM$ and $CJ = DN$. Applying the Hinge theorem and its converse introduced in Part 1, $IJ > MN$ if and only if $\angle ACB > \angle ADB$. Thus comparing $\angle ACB$ and $\angle ADB$ is equivalent to comparing IJ and MN. Since it is straightforward to compare the lengths of two line segments using a pair of compasses, the problem is solved.

Now that we have successfully found the best seats, there are still a few questions unanswered. What if AB is parallel to L? In that case, we cannot extend AB to meet L. What if A and B are not on the same side of L? As always, these questions are left as exercises for the readers.

Your lips uncontrollably curl to form a smile after you have found the perfect seats in the cinema. You check your pockets, and sure enough, the engagement ring is there. Your clothes are spotless. Everything is ready, yet there is still one thing missing. Where do you find a girlfriend?

Answer to "Try it out!" on Page 5:

$\angle APB = \angle CPD$ (common $\angle$)

$\angle ABP = \angle CDP$ (ext. $\angle$, cyclic quad.)

$\Delta APB \sim \Delta CPD$ (AA)

$\frac{AP}{CP} = \frac{BP}{DP}$ (corr. sides, $\sim\Delta$s)

$AP \cdot DP = BP \cdot CP$

Do you know?

The expression "if and only if" is a logical connective between statements. It is represented by the symbol "$\Leftrightarrow$" or the abbreviation "iff". The phrase "P if and only if Q" would imply that Q is necessary and sufficient for P. Making it easier to understand, the phrase would mean that both of the statements "if P, then Q" and "if Q, then P" hold true. The truth table of $P \Leftrightarrow Q$ is as follows:

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

It has been proved in Part 1 that $IJ > MN$ if $\angle ACB > \angle ADB$ and $\angle ACB > \angle ADB$ if $IJ > MN$. Therefore, we can conclude that $IJ > MN$ if and only if $\angle ACB > \angle ADB$.