# Formalizing May's Theorem

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#### Abstract

This report presents a formalization of May's theorem in the proof assistant Coq. It describes how the theorem statement is first translated into Coq definitions, and how it is subsequently proved. Various aspects of the proof and related work are discussed. To the best of the author's knowledge, this project is the first documented attempt in mechanizing May's Theorem.

# 1 Introduction

In 1952, Kenneth May published a mathematical theorem on social choice theory, which establishes a set of necessary and sufficient conditions for simple majority voting [13]. This result is now known as **May's theorem**. Though not as famous as Arrow's theorem [2] and the Gibbard-Satterthwaite theorem [9], May's theorem is still considered to be one of the "minor classics" in voting theory [4]. While the other above theorems in economics have been formalized in proof assistants [15][18], May's theorem has never been mechanically proved.

In this report, I present the first mechanized proof of May's theorem, which I implemented in **Coq** [5]. Proving the theorem in Coq not only increments the library of all proved theorems, but also provides insights in mechanizing results from social choice theory in a type theory based proof assistant. The structure of the Coq proof differs from the conventional proof sketch, and this report explains various subtle details of it.

In this document, I first present the high-level statement of the theorem in Section 2. In Section 3, I then elaborate how this statement is translated into Coq definitions. Section 4 and 5 then detail the structure of the Coq proof in the if direction and only if direction, respectively. In Section 6, I consider the nature of the proof itself by discussing its usefulness, correctness, and the extra extensionality axiom the proof relies on. Section 7 concludes the report and examines various future directions.

It should be noted that this document only mentions a small but important subset of the lemmas that constitute the main backbone of the Coq proof. A large number of relatively minor lemmas that the proof uses are omitted from this report for the sake of brevity. Although many of these lemmas appear straightforward at first glance, their mechanical proofs are quite convoluted and require much case analysis or extensive proof by induction. One example is the following lemma which states if two distinct elements, say x and y, are members of a list containing no duplicates, the list can be expressed as the concatenation of five lists, two of which are singleton lists containing x and y, respectively<sup>12</sup>:

<sup>&</sup>lt;sup>1</sup>The Coq keyword Lemma allows us to write a proposition for which its proof is then built using tactics.

<sup>&</sup>lt;sup>2</sup>NoDup 1 expresses that the list 1 contains no duplicates.

```
Lemma two_elements_exist {A:Type} (1:list A) x y `{!NoDup 1} `{x \neq y}: In x 1 \rightarrow In y 1 \rightarrow \frac{\pi}{11} 12 13, \\
1=\frac{1++[x]++12++[y]++13}{11} \rightarrow \frac{1=1++[y]++12++[x]++13}{11}.
```

# 2 May's Theorem

We now establish certain definitions following social choice theory nomenclature.

Consider an election with exactly two candidates x and y, and a finite set of voters. Each voter can cast a vote, called a **preference**, indicating that they either prefer x over y, y over x, or that they are indifferent between x and y.

A **social choice function** is a function that maps each possible list of preferences to a unique result, either x wins, y wins, or there is a tie between the two candidates.

Not all social functions are what we would perceive to be fair. May identifies several properties.

A social choice function is **anonymous** if it is a symmetric function of its arguments. Each voter is treated the same by the social choice function, and swapping the preference of any two voters will yield the same election result.

A social choice function is **neutral** if flipping the preference of each voter will also flip the function's result. (Here, an indifference preference remains unchanged after a flip). Both candidates are treated equally by the social choice function and swapping the names of the candidates will not affect the final result.

A social choice function is **monotone** if for any list of preferences where the result is indifferent or favorable to x, changing any voter's preference in a positive way towards x results in x winning the election. (Here, changing a preference in a positive way towards x means changing a y vote to an indifference one, a y to a x, or an indifference vote to a x.) This means the social choice function responds to changes of individual preferences in a "positive" manner.

Suppose the number of people who vote for x is a and the number of people who vote for y is b. The **simple majority function** is the social choice function that decides x wins if a > b, y wins if b > a, or that the two candidates tie otherwise.

May's theorem states that a social choice function is anonymous, neutral, and monotone if and only if it is the simple majority function. This means that anonymity, neutrality, and montonicity form a set of necessary and sufficient conditions for a social choice function to be the simple majority function.

# 3 Definitions and assumptions in Coq

In my proof, I used the Finite typeclass, from the library coq-std++ [1], to express the finite set of voters being considered<sup>3</sup>:

Context `{Finite voter}.

As shown below, this typeclass<sup>4</sup> provides three fields for the voter type: the field enum

<sup>&</sup>lt;sup>3</sup>The Coq keyword Context declares variables in the context of a section. In this case, the type voter belonging to the typeclass Finite.

<sup>&</sup>lt;sup>4</sup>Typeclasses are defined with the Coq keyword Record.

voter is a list containing all the voters, NoDup\_enum voter is a proposition stating that the list contains no duplicates, and elem\_of\_enum voter is a proposition stating that every term of the type voter is an element of the list. All three fields turn out to be equally important and are used extensively throughout the mechanical proof.

```
Record Finite (A : Type) (EqDecision0 : EqDecision A) : Type :=
Build_Finite { enum : list A;
   NoDup_enum : NoDup enum;
   elem_of_enum : ∀ x : A, x ∈ enum }.
```

Because whenever we want to reason about the two candidates (whether it is the preference of a voter or the result of the social choice function), we always have a third case where they tie, it is convenient to express the candidate type as option bool. Assuming the candidates are x and y, this allows us to represent x being preferred over y by Some true, y being preferred by Some false, and a tie with None. The use of bool also allows us to reason about the duality of preferences with simple bool functions like negb. The type for preferences and social\_choice\_function are simple function types<sup>56</sup>:

```
Definition preferences := voter → candidate.

Definition social_choice_function := preferences → candidate.
```

The anonymous property is expressed via a swap function, which swaps the preferences of two voters, while leaving the rest unchanged:

```
Definition swap (v1 v2:voter) (p:preferences) :=
  fun v =>
    if bool_decide (v=v1) then p v2
    else if bool_decide (v=v2) then p v1
    else p v.
Definition anonymous (scf:social_choice_function):=
    ∀ p v1 v2, scf p = scf (swap v1 v2 p).
```

The neutral property is expressed via a flip and flip\_vote function. The former flips a single candidate term, while the latter expresses the flipped version of the preferences of all voters:

```
Definition flip cand :=
   match cand with
   | Some b => Some (negb b)
   | None => None
   end.
Definition flip_vote (p:preferences) :=
   fun v:voter=> flip (p v).
Definition neutral (scf:social_choice_function) :=
    ∀ p, scf p = flip (scf (flip_vote p)).
```

The monotone property is expressed with an update function that updates the preference of a single voter while leaving the rest unchanged:

<sup>&</sup>lt;sup>5</sup>The Coq keyword Definition binds a term to a variable name.

<sup>&</sup>lt;sup>6</sup>The text description of the theorem talks about a list of preferences, but in Coq, it is simpler to model the preferences of voters as a function type.

```
Definition update v i (p:preferences) :=
   fun v':voter =>
      if bool_decide(v'=v) then i else p v'.
Definition monotone (scf:social_choice_function) :=
   ∀ p , (scf p = Some true ∨ scf p = None)→
      (∀ v, p v = Some false → scf (update v None p) = Some true) ∧
      (∀ v, p v = Some false → scf (update v (Some true) p) = Some true) ∧
      (∀ v, p v = None → scf (update v (Some true) p) = Some true).
```

After defining count, count\_helper, and various predicate functions for candidates like is\_some\_true, we can express the majority\_election rule as such:

```
Definition majority_election (p:preferences) :=
  let true_num := count is_some_true p in
  let false_num := count is_some_false p in
  if bool_decide (false_num < true_num ) then Some true
  else if bool_decide (true_num < false_num) then Some false
  else None.</pre>
```

Finally, one can express the main mays\_thm with the following proposition<sup>7</sup>:

```
Theorem mays_thm scf:
```

```
anonymous \operatorname{scf} \wedge \operatorname{neutral} \operatorname{scf} \wedge \operatorname{monotone} \operatorname{scf} \leftrightarrow \operatorname{scf} = \operatorname{majority\_election} .
```

One must also mention that the mechanical proof requires an additional axiom that is outside Coq's usual type-based calculus<sup>8</sup>. Specifically, we include the axiom of functional extensionality. The reason for this inclusion is discussed in subsection 6.4.

```
Axiom functional_extensionality {A B} (f g : A \rightarrow B): (\forall x, f x = g x) \rightarrow f = g.
```

#### 4 If direction

For the if direction of the theorem, we prove that the simple majority voting system is anonymous, neutral, and monotone.

## 4.1 Anonymity

The anonymous property of the simple majority function is mainly supported by the following invariant on swap<sup>9</sup>:

```
Lemma swap_invariant_count p f l v1 v2 `{!NoDup l}:
In v1 l \rightarrow In v2 l \rightarrow count_helper f p l = count_helper f (swap v1 v2 p) l.
```

This lemma states that given a subset of voters, swapping the preferences of any two voters within the subset does not change the overall number of each type of preference. In

<sup>&</sup>lt;sup>7</sup>The Coq keyword Theorem works the same as Lemma.

<sup>&</sup>lt;sup>8</sup>The Coq keyword Axiom extends the environment with an axiom.

<sup>&</sup>lt;sup>9</sup>The argument f ranges over predicate functions for candidates, like is\_some\_false.

fact this lemma relies on another similar one, which asserts that swapping the preferences of any two voters **not** within the subset also does not change the frequency of the types of preferences:

```
Lemma swap_not_in_list p f v1 v2 l:
    ~In v1 l → ~In v2 l → count_helper f p l =
        count_helper f (swap v1 v2 p) l.
```

It is also possible that we choose to swap the preference of a voter with itself, in which case, the list of preferences remains unchanged:

```
Lemma swap_same v p: (swap v v p) = p.
```

### 4.2 Neutrality

The neutrality of the simple majority function is proved by showing that the frequency of Some true and Some false are swapped whenever we perform a flip\_vote on a list of preferences:

```
Lemma flip_reverse_count1 p 1:
    count_helper is_some_false (flip_vote p) l =
    count_helper is_some_true p l.
Lemma flip_reverse_count2 p l:
    count_helper is_some_true (flip_vote p) l =
    count_helper is_some_false p l.
```

# 4.3 Monotonicity

To show that the simple majority function is monotone, I proved a number of lemmas that specify how the number of each type of preference changes if we update a certain voter's preference for each possible case. Here I present one of them, which states that changing a Some false vote to a None decreases the Some false vote frequency by one:

```
Lemma update_count_lemma_1 v p l `{NoDup l}:
   p v = Some false → In v l →
   count_helper is_some_false p l =
      1 + count_helper is_some_false (update v None p) l.
```

There is also the case where if we are considering a subset of voters, and we update someone not in the subset, this does not affect the frequency of each type of preference within the original subset:

```
Lemma upgrade_not_in_list f p v cand l:

~In v l → count_helper f p l = count_helper f (update v cand p) l.
```

# 5 Only if direction

For the only if direction of the theorem, we prove that for any social choice function, denoted as scf, that is anonymous, neutral, and monotone, it is equivalent to the simple majority function.

Given our goal  $scf = majority\_election$ , by the axiom of functional extensionality, it suffices to prove that for any list of preferences, denoted as p, it is the case that  $scf p = majority\_election p$ . Subsequently, we perform case analysis of all the possible outcomes of scf p and  $majority\_election p$ . For the three cases where they match, the statement follows trivially. Thus, it suffices to show that if  $scf p \neq majority\_election p$ , we can achieve a contradiction where we can prove the False proposition.

There are six cases where scf p  $\neq$  majority\_election p:

Case number	Preference condition	scf p
1	<pre>count is_some_false p &lt; count is_some_true p</pre>	Some false
2	<pre>count is_some_false p &lt; count is_some_true p</pre>	None
3	<pre>count is_some_true p &lt; count is_some_false p</pre>	Some true
4	<pre>count is_some_true p &lt; count is_some_false p</pre>	None
5	<pre>count is_some_false p = count is_some_true p</pre>	Some false
6	<pre>count is_some_false p = count is_some_true p</pre>	Some true

Actually, half of the cases are redundant. For example if case number 3 holds, we can reduce it to a variation of case number 1. To see why this is the case, assume case number 3 holds, and consider the list of preferences with each element flipped, i.e. flip\_vote p, denoted as p'. By the two lemmas from subsection 4.2, we have count is\_some\_false p' < count is\_some\_true p'. In addition, by the neutrality condition of scf, we have scf p' = Some false. We now have a variation of case number 1 where we replace p with p'. In other words, we only need to find a contradiction for each of the following three cases:

Case number	Preference condition	scf p
1	<pre>count is_some_false p<count is_some_true="" p<="" pre=""></count></pre>	Some false
2	<pre>count is_some_false p<count is_some_true="" p<="" pre=""></count></pre>	None
3	<pre>count is_some_false p=count is_some_true p</pre>	Some false

For the rest of this section, unless otherwise specified, we use a and b to denote the number of Some true and Some false votes in the list of preferences p, respectively.

## 5.1 Case 1: a > b and scf p = Some false

Suppose a > b and scf p = Some false. Consider flip\_vote p, the list of all the preferences flipped, denoted as p1. The following three statements can be proved:

- Using proof by induction, the number of Some true and Some false votes in p1 are flipped with respect to that of p, i.e. count is\_some\_false p1 = a and count is\_some\_true p1 = b.
- 2. By case analysis, the preference of a voter in p is None if and only if their preference in p1 is None, i.e.  $\forall$  voter, p voter = None  $\leftrightarrow$  p1 voter = None.
- 3. By neutrality of scf, we have scf p1 = Some true.

We then construct the list of preferences p2, which is the same as p1 but we update the first (a - b) Some false votes in p1 to Some true via the function upgrade\_vote\_list<sup>10</sup>:

<sup>&</sup>lt;sup>10</sup>The Coq keyword Fixpoint is the same as the keyword Definition, except that it is used specifically for recursive definitions. It also performs additional checks that the defined function is total.

The following three statements then follow:

- Using proof by induction, the number of Some true and Some false votes in p2 are the same as that of p, i.e. count is\_some\_true p2 = a and count is\_some\_false p2 = b.
- 2. By showing that None votes are not changed from p1 to p2, it is the case that the preference of a voter in p is None if and only if their preference in p2 is None, i.e.  $\forall$  voter, p voter = None  $\leftrightarrow$  p2 voter = None.
- 3. By monotonicity of scf and the following lemma, we have scf p2 = Some true:

```
Lemma upgrade_vote_list_monotone scf p 1:
   monotone scf →
   scf p = Some true →
   scf (upgrade_vote_list p 1) = Some true.
```

Lastly, after defining a function for swapping a list of pairs of preferences, we show that there exists a simple list of swaps that enables us to transform p2 to p:

```
Fixpoint swaps p 1:=
  match 1 with
  | [] => p
  | (x,y)::t1 => swap x y (swaps p t1)
  end.
Lemma same_true_num_implies_swappable p p':
  (∀ x : voter, p x = None ↔ p' x = None) →
  count is_some_true p = count is_some_true p' →
  ∃ 1 , p = swaps p' 1.
```

In fact, this list can be constructed easily: it is the result of zipping the list of voters who voted Some true in p and Some false in p2, and the list of voters who voted the other way round, as highlighted by the following function:

```
Definition count_true_same_swap_list_helper p p' 1:=
  let 11:= left_true_right_false p p' l in
  let 12:= left_false_right_true p p' l in
  zip 11 12.
```

Various properties of this list have to be proved. For example, one has to prove that the two lists are of the same length, so no element is dropped during the zip process:

```
Lemma count_true_difference_relation p p' l `{!NoDup l}:
    (∀ x, p x = None ↔ p' x = None) →
    count_helper is_some_true p l = count_helper is_some_true p' l →
    length (left_true_right_false p p' l) =
        length (left_false_right_true p p' l).
```

Lastly, by the anonymity property of scf, we then have scf p = Some true. However, we started with the assumption that scf p = Some false, and thus a contradiction is achieved.

## 5.2 Case 2: a > b and scf p = None

In this case, where a > b and scf p = None, the proof is similar to that of case number 1. However, during the construction of p2, we need a different invariant to prove that scf p2 = Some true. Specifically, we use the following lemma together with the fact that the number of voters to be updated is non-zero (since a - b > 0):

```
Lemma upgrade_vote_list_monotone_weak scf p 1:
  monotone scf → (scf p = Some true ∨ scf p = None) →
  (scf (upgrade_vote_list p 1) = Some true ∨
      scf (upgrade_vote_list p 1) = None).
```

### 5.3 Case 3: a = b and scf p = Some false

In this case, where a = b and scf p = Some false, the entire proof for case number 1 almost works perfectly here for us to achieve a contradiction as well. The main difference is that we need not update any voters to produce p2 from p1 (as a - b = 0), i.e. p2 = p1.

### 6 Discussion

## 6.1 Why is this formalization useful?

The first obvious answer is that formalizing the proof of a theorem allows us to accept it as fact with more certainty. Humans make mistakes, and it is not uncommon to hear mathematical proofs widely accepted by the community are found to contain minor gaps and inaccuracies afterwards [8][11]. While it is unlikely for May's theorem to be false, formalizing it enables us to ensure we do not miss any edge cases in high-level proof sketches.

This project also enables one to gain a precise understanding of the technique used within the proof. Defining and proving properties of each step rigorously allows the programmer to gain deeper insights into how and why a proof works. In particular, from a more personal perspective, the first proof I wrote relies on the axiom of excluded middle, which is an axiom from classical logic that is not included in Coq's intuitionistic logic system. Only during a thorough review of the code did I realize that the proof can be rewritten slightly to circumvent using the axiom, and thus allowing my most recent proof to be completely constructive.

Nonetheless, I believe whether a project is useful or not is sometimes not the best way to justify its value. I mainly pursued working on this project simply because I was interested in it. To quote Benthem Jutting's PhD thesis [12]:

"A further motive, for the author, was that the work involved in the project appealed to him."

#### 6.2 Is this theorem not obvious?

It might be true that most intermediate steps of the proof are not difficult to understand. However this is not equivalent to saying that the theorem is obvious or that the project itself is trivial.

Firstly, as mentioned at the beginning, though most intermediate steps might be easy to comprehend, the proof of those steps might be long and tedious. When reading a pen-and-paper proof, we automatically infer various properties implicitly, but in a proof assistant, everything must be stated and proved explicitly, e.g. whether a list has no duplicates, whether an element is contained (or not contained) within a list.

In addition, without knowing the trick beforehand, it is not exactly straightforward how one can prove May's theorem, especially for the only if direction. Perhaps, it might be more accurate to describe the proof to be elegantly short as opposed to describing the theorem as trivial.

### 6.3 Is this proof correct?

There are two potential sources of errors which might lead to the proof being incorrect. Luckily, both are unlikely.

Firstly, the Coq proof itself might not be sound, meaning that we can trace the error back to a bug in the kernel of the Coq proof assistant. Coq is based on the Calculus of Inductive Constructions [16], which is a reliable and well-understood type theory. Coq also satisfies the de Bruijn criterion [3], meaning it generates proof terms that can be verified by an independent and relatively small kernel. As a powerful verification tool that has been maintained for more than thirty years, one can safely trust and assume the correctness of Coq's kernel without losing much sleep.

Another source of error might be due to the definitions stating a completely different theorem instead of that of May's. To reduce the likelihood of this mistake, definitions are written as clearly as possible and are written to reflect the original texts closely.

There is one part where the Coq proof differs from the original paper. In May's paper, there is actually a fourth property in the set of sufficient and necessary properties for simple majority voting, called **decisiveness**, which states the function must be defined and single-valued for all possible lists of preferences. This, however, is exactly the definition for a function in mathematics anyway! As so, many subsequent papers on the theorem omitted this property in the statement [7], which I also followed suit.

# 6.4 Is the axiom of extensionality necessary?

Recall that the mechanical proof assumes the axiom of extensionality, which is not part of Coq's standard library. It is used because when we assert that two social choice functions are equivalent, we implicitly mean that the functions agree on every possible input.

One can circumvent the need for the axiom by redefining May's theorem slightly:

Theorem mays\_thm2 scf:

```
anonymous scf \land neutral scf \land monotone scf \leftrightarrow \forall p, scf p = majority_election p.
```

I still stuck with the first definition (see Section 3), since I think that definition of equality is clearer, as justified by Subsection 6.3.

## 7 Conclusion and future directions

In this report, I showed how May's theorem is formalized in the Coq proof assistant. I discussed several aspects of the Coq proof, including the translation from the theorem statement, various lemmas used in the proof, and its correctness.

Since the original publication, various others have extended May's theorem in multiple ways, e.g. when there is an infinite number of voters [7], or when there are more than two candidates [10]. A possible extension is to formalize those generalized theorems in Coq as well.

Being the first formalization of May's theorem, this Coq proof establishes another step into formalizing fundamental results in social choice theory. Another extension would be to continue formalizing other interesting theorems in this area, such as the median voter theorem [6], the McKelvey-Schofield chaos theorem [14], and Sen's possibility theorem [17]. It might also be worthwhile to develop a library containing voting-related primitives and lemmas for formalizing similar theorems.

### References

- [1] Coq-std++. https://gitlab.mpi-sws.org/iris/stdpp, 2013.
- [2] Arrow, K. J. A difficulty in the concept of social welfare. *Journal of Political Economy* 58, 4 (1950), 328–346.
- [3] BARENDREGT, H. H., AND WIEDIJK, F. The challenge of computer mathematics. *Philosophical transactions. Series A, Mathematical, physical, and engineering sciences* 363 (11 2005), 2351–73; discussion 2374.
- [4] BARRY, B. M., AND HARDIN, R. Rational man and irrational society?: An introduction and sourcebook.
- [5] BERTOT, Y., AND CASTÉRAN, P. Interactive theorem proving and program development: Coq'Art: the calculus of inductive constructions. Springer Science & Business Media, 2013.
- [6] Black, D. On the rationale of group decision-making. *Journal of Political Economy* 56, 1 (1948), 23–34.
- [7] FEY, M. May's theorem with an infinite population. Social Choice and Welfare 23 (2004), 275–293.
- [8] FLEURIOT, J. D., AND PAULSON, L. C. Proving Newton's propositio kepleriana using geometry and nonstandard analysis in Isabelle. In *Automated Deduction in Geometry* (Berlin, Heidelberg, 1999), Springer Berlin Heidelberg, pp. 47–66.
- [9] GIBBARD, A. Manipulation of voting schemes: A general result. *Econometrica* 41, 4 (1973), 587–601.
- [10] GOODIN, R. E., AND LIST, C. A conditional defense of plurality rule: Generalizing may's theorem in a restricted informational environment. American Journal of Political Science 50, 4 (2006), 940–949.

- [11] GOUËZEL, S., AND SHCHUR, V. A corrected quantitative version of the Morse lemma. *Journal of Functional Analysis* 277, 4 (2019), 1258–1268.
- [12] L.S. VAN BENTHEM JUTTING, L. Checking Landau's "Grundlagen" in the Automath system. Proefschrift Technische Hogeschool Eindhoven.
- [13] MAY, K. O. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica* 20, 4 (1952), 680–684.
- [14] MCKELVEY, R. D. Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory* 12, 3 (1976), 472–482.
- [15] NIPKOW, T. Social Choice Theory in HOL: Arrow and Gibbard-Satterthwaite. J. Automated Reasoning 43 (2009), 289–304.
- [16] PAULIN-MOHRING, C. Introduction to the Calculus of Inductive Constructions. In All about Proofs, Proofs for All, B. W. Paleo and D. Delahaye, Eds., vol. 55 of Studies in Logic (Mathematical logic and foundations). College Publications, Jan. 2015.
- [17] SEN, A. K. A possibility theorem on majority decisions. *Econometrica 34*, 2 (1966), 491–499.
- [18] Wiedlijk, F. Formalizing Arrow's theorem. Sadhana 34 (03 2009), 193–220.